Computing dynamic optimal mechanisms when hidden types are Markov and controlled by hidden actions

Kenichi Fukushima
University of Wisconsin, Madison

Yuichiro Waki
University of Queensland

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This note documents how the main theoretical results in Fukushima and Waki (2011) extend to a richer setting where the agent can influence the evolution of his hidden type θ_t through a hidden action y_t. An example of such a setting is one where θ_t represents a hidden stock of wealth or human capital and y_t is a hidden investment. The notation follows Fukushima and Waki (2011) unless otherwise indicated.

In each period the agent draws a type $\theta_t \in \Theta$ and sends a report $r_t \in \Theta$ to the planner. The planner chooses an outcome $x_t \in X$ and recommends an action $y_t \in Y$ given the agent's history of reports. The agent then chooses an action $y'_t \in Y$ which may or may not equal y_t. We assume Y is a finite set with cardinality M.

If the agent's current type is θ_t and he chooses action y_t, his next period type θ_{t+1} is drawn from the density $\pi(\cdot|\theta_t, y_t) > 0$. The initial distribution is $\pi(\cdot|\theta_{-1}, y_{-1})$ where (θ_{-1}, y_{-1}) is publicly known. We let $\mathbf{Y}$ denote the set of function sequences $\mathbf{y} = \{y_t\}_{t=0}^{\infty}$, $y_t : \Theta^{t+1} \to Y$ for each t, and write

$$\Pr(\theta^t|\theta_{-1}, y_{-1}, \mathbf{y}) = \pi(\theta_t|\theta_{t-1}, y_{t-1}(\theta^{t-1})) \times \cdots \times \pi(\theta_1|\theta_0, y_0(\theta^0)) \times \pi(\theta_0|\theta_{-1}, y_{-1}).$$

We also let $\mathbf{y}|_{\theta^{t-1}} = \{y_{t+s}(\theta^{t-1}, \cdot)\}_{s=0}^{\infty}$ denote the continuation of $\mathbf{y}$ after θ^{t-1}.

An allocation is then a sequence $(\mathbf{x}, \mathbf{y}) = \{x_t, y_t\}_{t=0}^{\infty}$, where $x_t : \Theta^{t+1} \to X$ and $y_t : \Theta^{t+1} \to Y$ for each t. We do not introduce randomizations to keep the notation simple, although doing so is quite straightforward and useful for computations (as it helps obtain convexity).

If allocation $(\mathbf{x}, \mathbf{y})$ takes place, that is, if after each shock history θ^t the outcome $x_t(\theta^t)$ occurs and the agent chooses $y_t(\theta^t)$, the agent obtains lifetime utility:

$$U(\mathbf{x}, \mathbf{y}; \theta_{-1}, y_{-1}) = \sum_{t=0}^{\infty} \sum_{\theta^t} \beta^t u(x_t(\theta^t), y_t(\theta^t); \theta_t) \Pr(\theta^t|\theta_{-1}, y_{-1}, \mathbf{y})$$

and the planner incurs cost:

$$C(\mathbf{x}, \mathbf{y}; \theta_{-1}, y_{-1}) = \sum_{t=0}^{\infty} \sum_{\theta^t} q^t c(x_t(\theta^t)) \Pr(\theta^t|\theta_{-1}, y_{-1}, \mathbf{y}).$$

An allocation $(\mathbf{x}, \mathbf{y})$ is therefore incentive compatible if

$$U(\mathbf{x}, \mathbf{y}; \theta_{-1}, y_{-1}) \geq U(\mathbf{x} \circ \mathbf{r}, \mathbf{y}'; \theta_{-1}, y_{-1}), \quad \forall (\mathbf{r}, \mathbf{y}') \in \mathbf{R} \times \mathbf{Y} \tag{1}$$

and satisfies promise keeping if

$$U(\mathbf{x}, \mathbf{y}; \theta_{-1}, y_{-1}) \geq U_0. \tag{2}$$

The planning problem starting from $(\theta_{-1}, y_{-1}, U_0)$ is to minimize $C(\mathbf{x}, \mathbf{y}; \theta_{-1}, y_{-1})$ by choice of $(\mathbf{x}, \mathbf{y})$ subject to incentive compatibility and promise keeping.

We have the following analog of Lemma 1:

Lemma A1. *An allocation* $(\mathbf{x}, \mathbf{y})$ *is incentive compatible if and only if*

$$u(x_t(\theta^t), y_t(\theta^t); \theta_t) + \beta U_{t+1}(\theta^t; \theta_t, y_t(\theta^t)) \geq u(x_t(\theta^{t-1}, \theta_t'), y_t'; \theta_t) + \beta U_{t+1}(\theta^{t-1}, \theta_t'; \theta_t, y_t') \tag{3}$$

for all t, θ^{t-1}, θ_t, θ_t', *and* y_t', *where*

$$U_t(\theta^{t-1}; \theta_-, y_-) = \sum_{s=t}^{\infty} \sum_{\theta_t^s} \beta^{s-t} u(x_s(\theta^{t-1}, \theta_t^s), y_s(\theta^{t-1}, \theta_t^s); \theta_s) \Pr(\theta_t^s | \theta_-, y_-, \mathbf{y}|_{\theta^{t-1}}).$$

Proof. The only if part is clear. So let $(\mathbf{x}, \mathbf{y})$ satisfy (3) and fix $(\mathbf{r}, \mathbf{y}') \in \mathbf{R} \times \mathbf{Y}$. For each t, define $\mathbf{r}|^t$ and $\mathbf{y}'|^t$ by $(r|_s^t(\theta^s), y'|_s^t(\theta^s)) = (r_s(\theta^s), y_s'(\theta^s))$ for all $s \leq t$ and θ^s, and $(r|_s^t(\theta^s), y'|_s^t(\theta^s)) = (\theta_s, y_s(r^t(\theta^t), \theta_{t+1}^s))$ for all $s \geq t+1$ and θ^s. I.e., $(\mathbf{r}|^t, \mathbf{y}'|^t)$ follows $(\mathbf{r}, \mathbf{y}')$ until period t and then reverts back to truth-telling and obedience from $t+1$. Applying (3) inductively we have $U(\mathbf{x}, \mathbf{y}; \theta_{-1}, y_{-1}) \geq U(\mathbf{x} \circ \mathbf{r}|^0, \mathbf{y}'|^0; \theta_{-1}, y_{-1}) \geq U(\mathbf{x} \circ \mathbf{r}|^1, \mathbf{y}'|^1; \theta_{-1}, y_{-1}) \geq \cdots \geq U(\mathbf{x} \circ \mathbf{r}|^t, \mathbf{y}'|^t; \theta_{-1}, y_{-1})$ for any t. Since u is bounded and $\beta \in (0,1)$, this implies:

$$U(\mathbf{x}, \mathbf{y}; \theta_{-1}, y_{-1}) \geq \lim_{t \to \infty} U(\mathbf{x} \circ \mathbf{r}|^t, \mathbf{y}'|^t; \theta_{-1}, y_{-1}) = U(\mathbf{x} \circ \mathbf{r}, \mathbf{y}'; \theta_{-1}, y_{-1}).$$

Hence $(\mathbf{x}, \mathbf{y})$ is incentive compatible. □

Notice here that the continuation utility profile $U_t(\theta^{t-1}; \cdot, \cdot)$ is a function of $(\theta_-, y_-) \in \Theta \times Y$, so a recursive formulation in the spirit of Fernandes and Phelan (2000) has $N \times M$ continuous state variables.

We say that π has an order K mixture representation if we can write:

$$\pi(\theta | \theta_-, y_-) = \sum_{k=1}^{K} p_k(\theta) w_k(\theta_-, y_-),$$

where $p : \Theta \to \mathbb{R}_+^K$ and $w : \Theta \times Y \to \mathbb{R}_+^K$ satisfy $\sum_{\theta \in \Theta} p_k(\theta) = 1$ for each k and $\sum_{k=1}^{K} w_k(\theta_-, y_-) = 1$ for each (θ_-, y_-).

Under this representation, we can define

$$a_t(\theta^{t-1}) = \sum_{\theta_t} \Bigg\{ u(x_t(\theta^t), y_t(\theta^t); \theta_t) + \beta \sum_{s=t+1}^{\infty} \sum_{\theta_{t+1}^s} \beta^{s-t-1} u(x_s(\theta^s), y_s(\theta^s); \theta_s) \Pr(\theta_{t+1}^s | \theta_t, y_t(\theta^t), \mathbf{y}|_{\theta^t}) \Bigg\} p(\theta_t) \tag{4}$$

and write

$$U_t(\theta^{t-1};\cdot,\cdot) = \sum_{k=1}^{K} a_{kt}(\theta^{t-1}) w_k(\cdot,\cdot).$$

This suggests that, by using a_t instead of U_t as an endogenous state variable, it should be possible to reduce the dimensionality from $N \times M$ to K.

Let us now write $a_t(\theta^{t-1};\mathbf{x},\mathbf{y})$ to describe the mapping from $(\mathbf{x},\mathbf{y})$ to $a_t(\theta^{t-1})$ defined by (4). Let us also write $a_0(\mathbf{x},\mathbf{y}) = a_0(\theta^{-1};\mathbf{x},\mathbf{y})$, as this is independent of θ_{-1}. We then define the auxiliary planning problem starting from $(\theta_{-1}, y_{-1}, a_0)$ as the problem of choosing $(\mathbf{x},\mathbf{y})$ to minimize $C(\mathbf{x},\mathbf{y};\theta_{-1},y_{-1})$ subject to incentive compatibility (1) and

$$a_0(\mathbf{x},\mathbf{y}) = a_0. \tag{5}$$

We let $A^* \subset V^K$ denote the set of a_0's for which the constraint set of this problem is non-empty (which is independent of (θ_{-1}, y_{-1})) and let $J^* : \Theta \times Y \times A^* \to \mathbb{R}$ denote the optimal value function. If

$$a_0^* \in \arg\min_{a_0 \in A^*} \; J^*(\theta_{-1}, y_{-1}, a_0) \quad \text{s.t.} \quad a_0 \cdot w(\theta_{-1}, y_{-1}) \geq U_0,$$

then a solution to the auxiliary planning problem starting from $(\theta_{-1}, y_{-1}, a_0^*)$ is a solution to the planning problem starting from $(\theta_{-1}, y_{-1}, U_0)$.

The analog of Lemma 2 is:

Lemma A2. *An allocation $(\mathbf{x},\mathbf{y})$ satisfies the constraints of the auxiliary planning problem (1) and (5) if and only if there exists $\mathbf{a} = \{a_t\}_{t=0}^{\infty}$, $a_t : \Theta^t \to A^*$, such that $(\mathbf{x},\mathbf{y},\mathbf{a})$ satisfies*

$$u(x_t(\theta^t), y_t(\theta^t); \theta_t) + \beta a_{t+1}(\theta^t) \cdot w(\theta_t, y_t(\theta^t)) \geq u(x_t(\theta^{t-1}, \theta_t'), y_t'; \theta_t) + \beta a_{t+1}(\theta^{t-1}, \theta_t') \cdot w(\theta_t, y_t') \tag{6}$$

$$a_t(\theta^{t-1}) = \sum_{\theta_t} \left\{ u(x_t(\theta^t), y_t(\theta^t); \theta_t) + \beta a_{t+1}(\theta^t) \cdot w(\theta_t, y_t(\theta^t)) \right\} p(\theta_t) \tag{7}$$

for all t, θ^t, θ_t', y_t', and $a_0(\theta^{-1}) = a_0$.

Proof. Virtually identical to that of Lemma 2. □

The analog of the B operator therefore maps $A \subset V^K$ into

$$B(A) = \{a \in V^K | \exists (x, y, a^+) \in F(a; A)\}$$

where $F(a; A)$ is the set of function triples $(x, y, a^+) : \Theta \to X \times Y \times A$ satisfying:

$$u(x(\theta), y(\theta); \theta) + \beta a^+(\theta) \cdot w(\theta, y(\theta)) \geq u(x(\theta'), y'; \theta) + \beta a^+(\theta') \cdot w(\theta, y'), \quad \forall \theta, \theta', y'$$
$$a = \sum_{\theta} \left\{ u(x(\theta), y(\theta); \theta) + \beta a^+(\theta) \cdot w(\theta, y(\theta)) \right\} p(\theta).$$

At this point it is useful to construct a particular incentive compatible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ as follows. First pick any $\bar{x} \in X$ and let $W : \Theta \to \mathbb{R}$ solve the Bellman equation:

$$W(\theta) = \max_{y \in Y} \left\{ u(\bar{x}, y; \theta) + \beta \sum_{\theta_+} W(\theta_+) \pi(\theta_+ | \theta, y) \right\}.$$

For each θ let $\bar{y}(\theta)$ solve the right hand side problem. Then set $\bar{x}_t(\theta^t) = \bar{x}$ and $\bar{y}_t(\theta^t) = \bar{y}(\theta_t)$ for each t and θ^t. We then have for any t and θ^{t-1}:

$$U_t(\theta^{t-1}; \theta_-, y_-) = \sum_{s=t}^{\infty} \sum_{\theta_t^s} \beta^{s-t} u(\bar{x}, \bar{y}(\theta_s); \theta_s) \Pr(\theta_t^s | \theta_-, y_-, \mathbf{y}|_{\theta^{t-1}}) = \sum_{\theta} W(\theta) \pi(\theta | \theta_-, y_-).$$

So for each t, θ^{t-1}, θ_t, θ_t', y_t':

$$u(\bar{x}, \bar{y}(\theta_t); \theta_t) + \beta U_{t+1}(\theta^t; \theta_t, \bar{y}(\theta_t)) \geq u(\bar{x}, y_t'; \theta_t) + \beta U_{t+1}(\theta^{t-1}, \theta_t'; \theta_t, y_t').$$

It follows from Lemma A1 that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is incentive compatible.

We have the following analog of Proposition 3:

Proposition A3. *A^* is a non-empty and compact set, and is the largest fixed point of B. If $A_0 \subset V^K$ is a compact set satisfying $A_0 \supset B(A_0) \supset A^*$ (one example being $A_0 = V^K$) then $B^n(A_0)$ is decreasing in n and $\cap_{n=0}^{\infty} B^n(A_0) = A^*$. If $A_0 \subset V^K$ satisfies $A^* \supset B(A_0) \supset A_0$ (one example being $A_0 = \{a_0(\bar{\mathbf{x}}, \bar{\mathbf{y}})\}$), then $B^n(A_0)$ is increasing in n and* $\mathrm{cl}(\cup_{n=0}^{\infty} B^n(A_0)) =$ *A^*.*

Proof. The analogs of Lemmas 5-8 follow from virtually identical arguments. Thus: (i) $A \subset V^K$, $A \subset B(A) \implies B(A) \subset A^*$, (ii) $B(A^*) = A^*$, (iii) $A \subset A' \subset V^K \implies B(A) \subset B(A')$, and (iv) A is compact $\implies B(A)$ is compact. Similarly for the first two parts of the proposition.

To prove the final part of the proposition, suppose $A_0 \subset B(A_0) \subset A^*$. Then from (ii), (iii), and the compactness of A^*, we know that $B^n(A_0)$ is increasing and $\mathrm{cl}(\cup_{n=0}^{\infty} B^n(A_0)) \subset A^*$. To prove $A^* \subset \mathrm{cl}(\cup_{n=0}^{\infty} B^n(A_0))$, pick any $a \in A^*$. We construct a sequence in $\cup_{n=0}^{\infty} B^n(A_0)$ that converges to a. For this, first pick another $a' \in A_0 (\subset A^*)$. By the definition of A^* there exist incentive compatible allocations $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}', \mathbf{y}')$ such that $a = a_0(\mathbf{x}, \mathbf{y})$ and $a' = a_0(\mathbf{x}', \mathbf{y}')$. Next for each $n \geq 1$, do the following. Define $\mathbf{x}^n = \{x_t^n\}_{t=0}^{\infty}$ by truncating $\mathbf{x}$ after n periods and appending $\mathbf{x}'$. Thus for $t > n$:

$$(x_0^n(\theta^0), ..., x_t^n(\theta^t)) = (x_0(\theta^0), ..., x_n(\theta^n), x_0'(\theta_{n+1}^{n+1}), ..., x_{t-n-1}'(\theta_{n+1}^t)).$$

And let

$$(\mathbf{r}^n, \mathbf{y}^n) \in \arg \max_{(\check{\mathbf{r}}, \check{\mathbf{y}}) \in \mathbf{R} \times \mathbf{Y}} U(\mathbf{x}^n \circ \check{\mathbf{r}}, \check{\mathbf{y}}).$$

Here, since $(\mathbf{x}', \mathbf{y}')$ is incentive compatible, we can assume without loss that for $t > n$, $r_t^n(\theta^t) = \theta_t$ and $y_t^n(\theta^t) = y_{t-n-1}'(\theta_{n+1}^t)$. Finally, let $(\hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n) = (\mathbf{x}^n \circ \mathbf{r}^n, \mathbf{y}^n)$. By construction, $(\hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n)$ is incentive compatible, $a_0(\hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n) \geq a_0(\mathbf{x}^n, \mathbf{y})$, and $a_{n+1}(\theta^n; \hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n) = a'$ for all θ^n.

We next show $a_0(\hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n) \in \cup_{n=0}^{\infty} B^n(A_0)$ for all n. From the incentive compatibility of $(\hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n)$ and $a_{n+1}(\theta^n; \hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n) = a'$ we obtain by induction $a_0(\hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n) \in B^{n+1}(\{a'\})$. This, (iii), and the fact that $B^n(A_0)$ is increasing in n then imply the result.

To verify $a_0(\hat{\mathbf{x}}^n, \hat{\mathbf{y}}^n) \to a$ as $n \to \infty$, we pick an arbitrary subsequence $\{a_0(\hat{\mathbf{x}}^{n'}, \hat{\mathbf{y}}^{n'})\}_{n'=1}^{\infty}$ and show that it has a further subsequence $\{a_0(\hat{\mathbf{x}}^{n''}, \hat{\mathbf{y}}^{n''})\}_{n''=1}^{\infty}$ that converges to a. Applying to $(\mathbf{r}^n, \mathbf{y}^n)$ the argument we applied to $\mathbf{r}^n$ in the proof of Proposition 3, we obtain a subindex n'' along which $(\mathbf{r}^{n''}, \mathbf{y}^{n''})$ converges to some $(\tilde{\mathbf{r}}, \tilde{\mathbf{y}})$. Also for each t we have $x_t^{n''} = x_t$ for $n'' \geq t$. This together with the boundedness of u implies $a_0(\hat{\mathbf{x}}^{n''}, \hat{\mathbf{y}}^{n''}) = a_0(\mathbf{x}^{n''} \circ \mathbf{r}^{n''}, \mathbf{y}^{n''}) \to a_0(\mathbf{x} \circ \tilde{\mathbf{r}}, \tilde{\mathbf{y}})$. Combining this with $a_0(\hat{\mathbf{x}}^{n''}, \hat{\mathbf{y}}^{n''}) \geq a_0(\mathbf{x}^n, \mathbf{y})$ and $a_0(\mathbf{x}^n, \mathbf{y}) \to a$, we obtain $a_0(\mathbf{x} \circ \tilde{\mathbf{r}}, \tilde{\mathbf{y}}) \geq a$. But the incentive compatibility of $(\mathbf{x}, \mathbf{y})$ implies $a_0(\mathbf{x} \circ \tilde{\mathbf{r}}, \tilde{\mathbf{y}}) \leq a_0(\mathbf{x}, \mathbf{y}) = a$, so $a_0(\mathbf{x} \circ \tilde{\mathbf{r}}, \tilde{\mathbf{y}}) = a$.

Now let $A_0 = \{a_0(\bar{\mathbf{x}}, \bar{\mathbf{y}})\}$. From the incentive compatibility of $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, (ii), and (iii), we have $B(A_0) \subset A^*$. To see $A_0 \subset B(A_0)$, observe that if we set $(x(\theta), y(\theta), a^+(\theta)) = (\bar{x}, \bar{y}(\theta), a_0(\bar{\mathbf{x}}, \bar{\mathbf{y}}))$ for each θ we have $(x, y, a^+) \in F(a_0(\bar{\mathbf{x}}, \bar{\mathbf{y}}); A_0)$. □

The analog of the T operator maps $J : \Theta \times Y \times A^* \to \mathbb{R}$ into $TJ : \Theta \times Y \times A^* \to \mathbb{R}$, defined as:

$$TJ(\theta_-, y_-, a) = \inf_{(x,y,a^+) \in F(a;A^*)} \sum_{\theta} \left\{ c(x(\theta)) + qJ(\theta, y(\theta), a^+(\theta)) \right\} \pi(\theta | \theta_-, y_-). \tag{8}$$

The analog of Proposition 4 is therefore:

Proposition A4. *J^* is a bounded lower semicontinuous function, and $||T^n J - J^*|| \to 0$ as $n \to \infty$ for any bounded $J : \Theta \times Y \times A^* \to \mathbb{R}$. There exists a function $g^* : \Theta \times Y \times A^* \to (X \times Y \times A^*)^{\Theta}$ which attains the infimum on the right hand side of (8) when $J = J^*$, and for any such g^* the allocation $(\mathbf{x}^*, \mathbf{y}^*)$ defined recursively by $(x_t^*(\theta^t), y_t^*(\theta^t), a_{t+1}^*(\theta^t)) = g^*(\theta_{t-1}, y_{t-1}^*(\theta^{t-1}), a_t^*(\theta^{t-1}))(\theta_t)$ solves the auxiliary planning problem starting from $(\theta_{-1}, y_{-1}, a_0^*(\theta^{-1}))$.*

Proof. Virtually identical to that of Proposition 4. □

References

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