A cautionary note on linear aggregation in macroeconomic models under the RINCE preferences★

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Abstract

Farmer (1990, Quarterly Journal of Economics) formulated the Risk-Neutral Constant Elasticity (RINCE) preferences and obtained a closed-form solution to the consumption-saving problem in the presence of idiosyncratic shocks to flow income and investment return. Both the value and policy functions are linear in wealth, and their linearity has been exploited to facilitate aggregation in macroeconomic models. However, Farmer's solution implicitly assumed that the natural borrowing limit never binds. A counterexample is provided herein in which the correct solution is nonlinear because the natural borrowing limit binds. To resurrect the linearity, one needs to restrict carefully the shock process of flow income and investment return so that the natural borrowing limit never binds. By doing so, however, one could throw away some important classes of problems with income shocks. Two models with temporary and persistent income shocks are used to show that the linear solution violates the natural borrowing limit with non-negligible probability in realistic settings.

Keywords: Consumption-saving problems, the RINCE preferences, Borrowing constraints, Aggregation
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1. Introduction

Consumption-saving problems in the presence of shocks to income and return on saving do not generally permit closed-form solutions. These problems have been central to heterogeneous-agents incomplete-market models in which households are hit by uninsurable, idiosyncratic shocks to their income or the rate of return on their saving (Bewley, 1983; Imrohoroglu, 1992; Huggett, 1993; Aiyagari, 1994). The lack of closed-form solutions has led researchers to use numerical methods to solve these problems. In general, there is no explicit aggregation theorem for decisions of heterogeneous households particularly when both shocks to their income and the rate of return are idiosyncratic.[1]

Farmer (1990) argued that a closed-form solution is available for a particular class of preferences, which he coined the RIsk-Neutral Constant Elasticity (RINCE) preferences. It is a special case of the Kreps-Porteus preferences (Kreps and Porteus, 1978, 1979) and assumes the risk neutrality but the intertemporal elasticity of substitution is finite and constant. He obtained the closed-form solution while allowing for an arbitrary joint shock process for income and for the rate of return on saving.[2]

An important property of this closed-form solution is that both the value and the policy functions of a decision-maker are linear in her wealth. The decision-maker's past actions affect the current value and policy functions only through her current wealth. All coefficients on wealth in these functions are determined exogenously by the stochastic process of real interest rates and the preference parameters. The linearity of a solution has been exploited to obtain aggregation in macroeconomics models since Gertler (1999).

This paper shows that there was an implicit assumption made by Farmer (1990) when he obtained the linear closed-form solution. The paper provides a counterexample in which this assumption is invalid. When the assumption is violated, the true solution is no longer linear in wealth, and linear aggregation fails.

The assumption is that the "natural" borrowing limit (Aiyagari, 1994) never binds; that is, the decision-maker never finds it optimal to borrow more than she can repay with certainty. When formulating the recursive version of the decision problem, Farmer (1990) only required that the decision-maker not leave any debt upon dying (i.e., in the terminal period), but imposed no borrowing constraints before the terminal period. His linear closed-form solution is the solution to this recursive problem.

However, when the aforementioned implicit assumption is violated, the closed-form solution in Farmer (1990) implies that the decision-maker accumulates so much debt that she cannot repay when her flow income turns out to be low. In other words, it is not a solution to the original, nonrecursive problem when the natural borrowing limit binds.[3] The counterexample in the present paper is a two-period example; in the second period, flow income is random and may become zero with strictly positive probability. Given that the second period value function is linear in wealth as described in Farmer (1990) and that no borrowing constraint is imposed in the first period, the decision-maker has an incentive to borrow in the first period. This incentive is because her future income is, on average, higher than the current cash-on-hand and because she has a consumption-smoothing motive. However, if her flow income in the second period turns out to be zero, she cannot repay debt while entertaining non-negative consumption.

Therefore, a valid recursive formulation requires some borrowing constraints not only in the terminal period but also in the periods beforehand; however, then the solution is no longer linear in wealth. Once we impose some borrowing constraints, whether natural or ad hoc, we have occasionally binding constraints in the decision problem, and these create kinks in the value and the policy functions. I demonstrate this point by adding a borrowing constraint to the above counterexample.

Losing the linearity means much. Since Gertler (1999), the RINCE preferences have been widely used in macroeconomic models because the linearity of the value and policy functions can be exploited to facilitate

[1] The only exception that I am aware of is Braun and Nakajima (2012). Assuming a homothetic utility function as well as a particular correlation structure between the income and the rate of return shocks, Braun and Nakajima establish an aggregation result.

[2] They are only required to take on non-negative real values.

[3] Another way to put it is that the recursive formulation in Farmer (1990) is incorrect when the implicit assumption is not satisfied.

aggregation in the presence of uninsurable, idiosyncratic shocks.[4] Given that all households face the same real interest rate process, total consumption and saving within a group of households that share the same RINCE preferences are linear in within-group aggregate wealth. This property makes the model tractable because we do not need to keep track of the within-group wealth distribution when computing aggregate variables. However, without linearity, such aggregation is impossible.

It is possible to preserve the linearity of a solution by imposing more restrictions on the problem structure so that the borrowing constraints never bind. One source of the problem is that Farmer (1990) allowed the flow income and the real interest rates to follow *any* non-negative joint stochastic process. This approach is too flexible, and one may be able to restrict the stochastic process and entertain the linearity of a solution while keeping the problem both realistic and interesting. Section 4 makes this point.

However, as I will demonstrate in Section 5, some important classes of problems cannot be analyzed while entertaining the linearity. For this purpose I use two models to evaluate how likely the linear solution in Farmer (1990) violated the natural borrowing limit. One model features a temporary income shock and an income buffer that is provided by either a formal or informal insurance arrangement. The other features a large income shock due to job loss that may become persistent because of a search and matching friction and a simple yet realistic unemployment insurance. In the former model, I prove an asymptotic result that the probability that the linear solution violates the natural borrowing limit in period t converges to one half as t goes to infinity. This result does not depend on the details of the model. Conversely, I demonstrate the possibility that the natural borrowing limit does not bind for a sufficiently long period when there is an income buffer that protects households from downside income risk. This result depends crucially on the distribution of the before-transfer income, and for some distributions, too many households need to be protected by the income buffer in order to prevent the natural borrowing limit from binding for a sufficiently long period. In the latter model, the probability of the linear solution violating the natural borrowing constraint is sizable, and that the result is robust to changes in key parameter values, such as the unemployment benefit duration and the replacement rate.

2. The stochastic decision problem in Farmer (1990)

A decision-maker lives from $t = 0$ to T and maximizes her lifetime utility evaluated in period 0, v_0, by choosing a consumption-saving plan, $\{c_t, a_{t+1}\}_{t=0}^{T}$, subject to the flow budget constraints. The flow budget constraints are given by:

$$c_t + a_{t+1} \leq R_t a_t + \omega_t, \quad \forall t = 0, \cdots, T, \tag{1}$$

where R_t denotes the gross real interest rate earned on the period $t-1$ saving a_t and ω_t denotes the flow income in period t. Both R_t and ω_t are allowed to follow any non-negative, joint stochastic process.

The initial and the terminal conditions for her saving $\{a_t\}$ are:

$$a_0 = \overline{a}_0 \tag{2}$$

and

$$a_{T+1} \geq 0. \tag{3}$$

The decision-maker's lifetime utility from $\{c_t\}$ follows a recursion:

$$v_t = w(c_t, \mathbb{E}_t v_{t+1}), \forall t = 0, \cdots, T-1, \quad v_T = w_T(c_T). \tag{4}$$

The decision-maker's problem is to maximize v_0 by choosing $\{c_t, a_{t+1}\}_{t=0}^{T}$ subject to the flow budget constraints (equation 1), the initial and the terminal condition for saving (equations 2 and 3), and the recursion for utility (equation 4), and the non-negativity constraint for consumption: $c_t \geq 0$, for all $t = 0, \cdots, T$.

[4]Examples include Fujiwara and Teranishi (2008), Carvalho, Ferrero, and Nechio (2016), Basso and Rachedi (2021) and Fujiwara, Hori, and Waki (2019). These papers and Gertler (1999) assume probabilistic aging to incorporate an overlapping generations structure in general equilibrium models, while maintaining tractability using age-group-specific aggregation.

Here the function w in equation (4) is a homogeneous function and is given by:

$$w(x, y) = (x^{\rho} + \beta y^{\rho})^{1/\rho}$$

with $\beta \in [0, \infty)$ and $\rho \in (-\infty, 0) \cup (0, 1]$.[5] The function w_T is simply $w_T(x) = x$. Therefore, the utility function here is a special case of the Kreps-Porteus preferences (Kreps and Porteus, 1978, 1979), parameterized later in Epstein and Zin (1989). It assumes the risk neutrality but the intertemporal elasticity of substitution is finite and constant. Hence it is coined a RIsk-Neutral Constant Elasticity (RINCE) preference.

Importantly, income, $\{\omega_t\}_{t=0}^{T}$, and return on saving, $\{R_t\}_{t=0}^{T}$, are subject to exogenous shocks. When both shocks are present, a closed-form solution does not exist for general preference specifications.

2.1. Characterization of the solution in Farmer (1990)

Farmer (1990) argued that the above problem permits a rather simple, closed-form solution that is linear in a certain measure of wealth. First define two functions, F and G, as

$$F(x) := \left(1 + \beta^{\frac{1}{1-\rho}} x^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}} \quad \text{and} \quad G(x) := \left(1 + \beta^{\frac{1}{1-\rho}} x^{\frac{\rho}{1-\rho}}\right)^{-1}.$$

Then define two stochastic processes $\{Q_t\}_{t=0}^{T}$ and $\{h_t\}_{t=0}^{T}$ recursively:

$$F(Q_T) = 1, \text{ and } Q_t = \mathbb{E}_t[R_{t+1} F(Q_{t+1})], \text{ for all } t = 0, 1, \cdots, T-1,$$

and

$$h_T = \omega_T, \text{ and } h_t = \omega_t + \mathbb{E}_t\left[h_{t+1} \frac{F(Q_{t+1})}{Q_t}\right], \text{ for all } t = 0, 1, \cdots, T-1.$$

Here $F(Q_{t+1})/Q_t$ is a stochastic discount factor and h_t can be interpreted as human wealth that equals the present discounted value of income from period t onwards. The above recursion does not involve endogenous variables. Therefore, these two stochastic processes are exogenously determined.

Farmer argued that the solution to the decision problem is written as

$$c_t = G(Q_t) W_t, \tag{5}$$
$$v_t = F(Q_t) W_t, \tag{6}$$

for all $t = 0, 1, ..., T$, where W_t denotes the beginning-of-period-t total wealth, defined as the sum of financial and human wealth, $W_t := R_t a_t + h_t$.

The solution characterized above has a nice property: the optimal consumption and saving are linear in wealth. This property is used to obtain aggregation results in some macroeconomic studies discussed later in Section 4.

2.2. Sketch of the proof in Farmer (1990)

Farmer (1990) provided a sketch of the proof based on a recursive formulation. For any t, any a, and any history of exogenous shocks up to period t, let

$$V_t(a) := F(Q_t)(R_t a + h_t), \tag{7}$$
$$C_t(a) := G(Q_t)(R_t a + h_t), \tag{8}$$

and

$$A'_t(a) := R_t a + \omega_t - C_t(a). \tag{9}$$

[5] Farmer (1990) also considered a Cobb-Douglas aggregator where ρ is taken to zero, but for notational simplicity I restrict attention to the $\rho \neq 0$ cases. When the planning horizon is infinite, β needs to be further restricted to be less than one.

Here the dependence of these functions on an exogenous shock history is kept implicit and represented by the subscript t. Then the sequence of value functions, $\{V_t\}_{t=0}^T$, and policy functions, $\{C_t, A'_t\}_{t=0}^T$, solves the following functional equations: in the terminal period T, for any a, and any history of exogenous shocks up to period T,

$$V_T(a) = \max_{c,a'} c$$

subject to the budget constraint, $c + a' \leq R_T a + \omega_T$, and the non-negativity constraint: $c, a' \geq 0$; and, for any $t = 0, 1, \cdots, T-1$, any a, and any history of exogenous shocks up to period t,

$$V_t(a) = \max_{c,a'} \{c^\rho + \beta (\mathbb{E}_t V_{t+1}(a'))^\rho\}^{1/\rho}$$

subject to the budget constraint, $c + a' \leq R_t a + \omega_t$ and the non-negativity constraint for consumption: $c_t \geq 0$.

The proof is based on the backward induction and the first-order condition of the above recursive problem.

3. A counterexample

What is the problem? The problem is that the above recursive formulation is incorrect. Only when the natural borrowing limit is slack in all periods and states is it correct. In this model, however, the natural borrowing limit can bind, and the solution proposed in Farmer (1990) is incorrect when it binds. The following example illustrates this point.

3.1. An example where Farmer's solution is incorrect

Consider the following two-period model, which is a special case of one presented in Farmer (1990) with $T = 1$. Set the parameters as follows: $\beta = 1$, $R_0 a_0 + \omega_0 < 0.5$, $R_1 = 1$ (with certainty), and $\omega_1 = 0$ with probability $\epsilon < 0.5$ and $\omega_1 = 1$ with probability $1 - \epsilon$. Let us calculate Farmer's solution. First, using the recursion for Q and h, we obtain:

$$Q_0 = \mathbb{E}_0[F(Q_1)] = 1,$$

and

$$h_0 = \omega_0 + \mathbb{E}_0 h_1 = \omega_0 + \mathbb{E}_0 \omega_1.$$

Consumption and value are then obtained using the policy and value functions as follows:

$$\begin{aligned} c_0 &= G(1)W_0 = \frac{W_0}{2}, \\ c_1 &= a_1 + \omega_1, \\ v_0 &= F(1)W_0 = 2^{\frac{1-\rho}{\rho}} W_0, \\ v_1 &= a_1 + \omega_1, \end{aligned}$$

where $W_0 = R_0 a_0 + h_0 = R_0 a_0 + \omega_0 + \mathbb{E}_0 \omega_1$. The optimal saving a_1 is, therefore,

$$\begin{aligned} a_1 &= R_0 a_0 + \omega_0 - c_0 \\ &= R_0 a_0 + \omega_0 - \frac{W_0}{2} \\ &= R_0 a_0 + \omega_0 - \frac{R_0 a_0 + \omega_0 + \mathbb{E}_0 \omega_1}{2} \\ &= \frac{1}{2}\{R_0 a_0 + \omega_0 - \mathbb{E}_0 \omega_1\}. \end{aligned}$$

Because $R_0 a_0 + \omega_0 < 0.5$ and $\mathbb{E}_0 \omega_1 = 1 - \epsilon > 0.5$, the last expression is negative. Hence, the optimal saving according to Farmer's solution is negative.

However, this cannot be a solution to the original, nonrecursive problem. In period 1, the decision-maker's flow income ω_1 may be zero with positive probability ϵ. In that state, with zero income, the decision-maker cannot repay her debt while entertaining non-negative consumption. Hence, the terminal condition $a_2 \geq 0$ must be violated in the zero-income state.

3.2. Diagnosis

What is wrong? In Farmer's recursive formulation, the household's choices are restricted only by the flow budget constraints, except for the terminal period in which saving must be non-negative. However, to ensure that the household does not leave any debt upon dying in all states, one must impose some borrowing constraints, such as the natural borrowing limit or an ad hoc borrowing limit (Aiyagari, 1994), which restricts the household's ability to borrow *before the terminal period*. When the borrowing constraint binds, the value function is no longer linear in a proposed measure of wealth and has a kink above which the borrowing constraint becomes slack.

The natural borrowing limit is the maximal amount of debt a decision-maker can repay with certainty without violating the non-negativity constraint for consumption. The natural borrowing limit in the two-period example is:

$$a_1 \geq -\min \frac{\omega_1}{R_1},$$

where the minimization on the right-hand side is over the set of all possible states in period 1. Because $R_1 = 1$ with certainty and $\min \omega_1 = 0$, the natural borrowing limit in the example is equivalent to $a_1 \geq 0$; that is, no borrowing is allowed. This constraint binds as far as $R_0 a_0 + \omega_0 < 0.5$, and it is slack for $R_0 a_0 + \omega_0 \geq 0.5$. Failing to take this constraint into account can leads us to an incorrect solution. The period-0 value and policy functions take the form as in equations (7)-(9) for $a_0 \geq (0.5 - \omega_0)/R_0$. For $a_0 < (0.5 - \omega_0)/R_0$, however, they are given by:

$$\begin{aligned} C_0(a) &= R_0 a_0 + \omega_0, \\ A_0'(a) &= 0, \end{aligned}$$

and

$$V_0(a) = ((R_0 a_0 + \omega_0)^\rho + (\mathbb{E}_0 \omega_1)^\rho)^{1/\rho} = ((R_0 a_0 + \omega_0)^\rho + (1-\epsilon)^\rho)^{1/\rho},$$

which is obviously nonlinear in a.

In incomplete-market models with income risk, if the utility function is a time-separable, expected utility with geometric discounting, the Inada condition ensures that the natural borrowing limit never binds: if it binds, the household's consumption becomes zero after some history of shocks, and it is suboptimal under the Inada condition.

Does the assumption of a CES preference aggregator imply the Inada condition? Although the preference aggregator is nonlinear in current consumption, the answer is no. This failure of the Inada condition is most clearly observed in the above two-period example. Substitute the the identity $v_1 = c_1$ into the preference aggregator in period 0 to obtain:

$$v_0 = \max_{c_0, a_0} \{c_0^\rho + \beta(\mathbb{E}_0 c_1)^\rho\}^{1/\rho}.$$

Hence, the decision-maker has an incentive to smooth average consumption over time, but when the expected future consumption is sufficiently high, the decision-maker does not mind consumption (or the continuation value) becoming zero in some future states because of risk neutrality. When the expected future income is sufficiently high relative to the current cash-on-hand, the decision-maker wants to borrow as much as possible so that the natural borrowing limit binds.

3.3. *A correct recursive formulation with the natural borrowing limit*

It is possible to formulate the recursive problem while incorporating occasionally binding natural borrowing constraints because the maximal amount that can be repaid is also defined recursively. Define the stochastic process of natural borrowing limits, $\{a_{t+1}^{LB}\}_{t=0}^{T}$, as follows:

$$a_{T+1}^{LB} = 0$$

and

$$a_t^{LB} = \max \frac{a_{t+1}^{LB} - \omega_t}{R_t}, \quad \forall t = 0, 1, \cdots, T.$$

Note that for all t, though implicit, the left-hand side object a_t^{LB} is contingent on the history of shocks up to period $t-1$ (inclusive). Each variable in the maximand on the right-hand side shares the same history up to period $t-1$ with the left-hand side variable, but also depends on the shock realization in period t. The maximization is over the set of all the possible shock realizations in period t. This variable a_t^{LB} is forward-looking but determined exogenously. Hence, it can be computed separately from endogenous variables.

Thus, the correct recursive formulation is given as follows. First, in the terminal period T, the problem is identical to that given in Farmer (1990). For any $t \leq T-1$, any a, and any history of exogenous shocks up to period t,

$$V_t(a) = \max_{c,a'} \{c^\rho + \beta\left(\mathbb{E}_t V_{t+1}(a')\right)^\rho\}^{1/\rho}$$

subject to the budget constraint, $c + a' \leq R_t a + \omega_t$, and the natural borrowing limit: $a' \geq a_t^{LB}$.

A solution to this problem is not as simple as given in Farmer (1990). The natural borrowing limit is an occasionally binding constraint, and the decision rule is necessarily nonlinear. In addition to nontractability, the solution may be unrealistic, because once the natural borrowing limit binds, consumption falls to zero permanently, which is not a desirable property if the model is used to match actual data.

4. Discussion

One way to resurrect the linearity of the solution is to impose additional restrictions so that the natural borrowing limit never binds. For example, one may assume away the possibility of a temporary, negative shock to income, which incentivizes households to borrow. Alternatively, one may instead assume that a decision-maker is sufficiently patient so that she finds it optimal to save, even if hit by the largest possible negative temporary income shock. With such restrictions, the solution in Farmer (1990) would be indeed correct.

These restrictions may not be as restrictive as they might appear. Gertler (1999) formulated a tractable, overlapping generations model using the RINCE preferences and probabilistic aging, in order to study the macroeconomic effects of fiscal policy and of social security. Young people age (or does not age) with a constant probability, while older people die with a constant probability. As far as the older person's labor income is sufficiently lower than the younger person's, and the older person's labor income does not increase much over time, nobody in the economy has incentives to borrow. In this case, the natural borrowing constraint never binds, and consumption and savings are linear in wealth. Because of the linearity of the decision rules, the aggregate equilibrium behavior of the model depends on the wealth distribution only through the total amount of wealth (the sum of capital stock and human wealth) in the economy and the fraction (or the amount) of wealth held by the young. This property facilitates equilibrium computation.[6]

However, it is worth noting that some interesting model specifications may be ruled out by imposing additional restrictions on the model. To demonstrate this point, I consider in the next section a model with a temporary, small income shock and a model with a possibly persistent, large income shock. These models

[6]The same modeling strategy is used in Fujiwara and Teranishi (2008), Carvalho, Ferrero, and Nechio (2016), Basso and Rachedi (2021), and Fujiwara, Hori, and Waki (2019) to introduce an overlapping generations structure into New Keynesian models.

are used to evaluate how likely it is that the linear solution in Farmer (1990) violates the natural borrowing limit.

5. Evaluating the relevance of the binding borrowing constraint

Now I use two models to evaluate how likely the natural borrowing constraint binds. To simplify the analysis, I make two key assumptions in both models. The first assumption is that $\beta R = 1$. This assumption simplifies the marginal propensity to consume out of wealth for two reasons. First, the real interest rates are deterministic and constant, and, therefore, both $\{Q_t\}$ and the marginal propensity to consume become deterministic. Second, when $\beta R = 1$, the marginal propensity to consume out of wealth is independent of the elasticity of intertemporal substitution. Indeed, under the assumption of $\beta R = 1$, the household's decision rule is identical to a standard quadratic utility framework.

The assumption of $\beta R = 1$ is appropriate when examining whether it is valid to assume that the borrowing constraint never binds. Consider a heterogeneous-agent incomplete-market macroeconomic model with idiosyncratic labor income risk, such as described in Aiyagari (1994) and Huggett (1993) and imagine that the households in the model have the common RINCE preferences. *Suppose* that the borrowing constraint never binds in a steady-state equilibrium of the model. Then, because the households are risk neutral and do not have precautionary saving motives, $\beta R = 1$ would hold in a steady-state equilibrium. I can examine whether such a steady-state equilibrium exists by analyzing how frequently the natural borrowing limit is actually violated when $\beta R = 1$. If the natural borrowing limit is frequently violated under the assumption of $\beta R = 1$, then it suggests that there is no steady-state equilibrium in which the natural borrowing limit is virtually slack. In other words, the natural borrowing limit must be imposed in such models.

For the second assumption, I consider an infinite horizon model by letting T tend toward infinity. Even under the first simplifying assumption of $\beta R = 1$, the marginal propensity to consume is time-dependent if T is finite. In contrast, the marginal propensity to consume becomes constant over time in an infinite horizon model. This property not only simplifies notation but also simplifies the savings dynamics.

In an infinite horizon model with $\beta R = 1$, the dynamics of savings have a simple expression because the marginal propensity to consume out of wealth is constant at $1-\beta$. Substituting $c_t = (1-\beta)(Ra_t + h_t)$ and $\beta R = 1$ into the budget constraint, I obtain the equation that governs the savings dynamics:

$$\Delta a_{t+1} := a_{t+1} - a_t = \beta\{\omega_t - (1-\beta)\mathbb{E}_t h_{t+1}\}. \tag{10}$$

Note that the right-most expression contains exogenous shocks and their expectations only. If, for example, the flow income ω_t follows a time-homogeneous Markov chain, the right-most expression also follows a time-homogeneous Markov chain. Moreover, the latter chain is easily calculable from the former.

It is worth emphasizing that my assumptions so far do not imply a negative drift for savings. In this sense, these assumptions are not pushing the average savings toward the natural borrowing limit. To observe this, take the unconditional expectations of both sides in equation (10) and obtain:

$$\mathbb{E}\Delta a_{t+1} = \beta(\mathbb{E}\omega_t - (1-\beta)\mathbb{E}h_{t+1}) = \beta\left\{\mathbb{E}\omega_t - (1-\beta)\sum_{s=t+1}^{\infty}\beta^{s-t-1}\mathbb{E}\omega_s\right\}. \tag{11}$$

As long as $\{\omega_t\}$ is a stationary process, the right-most expression is zero, because $\mathbb{E}\omega_t = \mathbb{E}\omega_s$ for any t and s. Hence, the savings stay constant *in expectations* and there is no negative drift for savings.

5.1. A model with temporary income loss

The first model is one with a transitory shock to income. I model a transitory shock as $\{\omega_t\}$ being a non-negative, independent and identically distributed (IID) process with a strictly positive variance. Let $\omega_{min} \geq 0$ be the lowest possible value that ω_t can take. Then the natural borrowing limit is given by $a_{LB} = -\beta\omega_{min}/(1-\beta)$.

Denoting the mean of ω_t by ω^e, equation (10) reduces to

$$\Delta a_{t+1} = \beta(\omega_t - \omega^e). \tag{12}$$

Because the variance of ω is strictly positive, the savings follows a random walk without drift. Then,

$$a_{t+1} = a_0 + \beta \sum_{s=0}^{t} (\omega_t - \omega^e), \tag{13}$$

and

$$var(a_{t+1}) = \beta^2 \sigma_\omega^2 \times (t+1), \tag{14}$$

where σ_ω^2 is the variance of ω_t.

First, I show that, asymptotically, the natural borrowing limit will be violated with a probability of one-half. The probability of the period-t savings violating the natural borrowing limit is given by:

$$\text{Prob}\,(a_{t+1} < a_{LB}) = \text{Prob}\left(\frac{a_{t+1} - a_0}{\beta\sigma_\omega\sqrt{t+1}} < \frac{a_{LB} - a_0}{\beta\sigma_\omega\sqrt{t+1}}\right). \tag{15}$$

By the central limit theorem, $(a_{t+1} - a_0)/\sqrt{t+1}$ converges in distribution to $N(0, \beta^2\sigma_\omega^2)$ as $t \to \infty$, and

$$\lim_{t\to\infty} \text{Prob}\left(\frac{a_{t+1} - a_0}{\beta\sigma_\omega\sqrt{t+1}} < \frac{a_{LB} - a_0}{\beta\sigma_\omega\sqrt{t+1}}\right) = \Phi\,(0) = 0.5, \tag{16}$$

where Φ denotes the cumulative distribution function for the standard normal distribution. This asymptotic result is strong: it is independent of the value of the natural borrowing limit or of the variance of ω or the initial asset level a_0.

Of course, for a finite t, the probability that the natural borrowing limit binds is dependent on these parameters. To observe this relationship, suppose that ω_t can be written as $\omega_t = \min\{\omega_{min}, \gamma_t\}$, where γ_t is a non-negative IID process with a strictly positive variance and with the smallest possible realization $\gamma_{min} < \omega_{\min}$. This assumption can be interpreted as follows. A household receives random flow income γ_t, but there is either a formal or informal insurance arrangement so that when the flow income is less than a threshold ω_{min}, the difference between the actual and threshold income is compensated by a transfer payment. In other words, there is an income buffer. The income after transfer equals to the lower bound ω_{min} with probability $\epsilon := \text{Prob}(\gamma_t \leq \omega_{min})$. The natural borrowing limit is the same as before and is given by $a_{LB} = -\beta\omega_{min}/(1-\beta)$.

Now imagine that the household keeps receiving the lowest possible income after a transfer payment, ω_{min}, and compute how many periods it takes to violate the natural borrowing limit. Let ω^e denote the mean of ω_t.[7] Then if the household keeps drawing ω_{min} from period 0 to t, its asset level is given by $a_{t+1} = a_0 + \beta(t+1)(\omega_{min} - \omega^e)$. This asset level is lower than a_{LB} if and only if

$$t + 1 > \frac{a_0 - a_{LB}}{\omega^e - \omega_{min}}. \tag{17}$$

Therefore, the larger the right-hand side expression, the longer it takes for the natural borrowing limit to bind with strictly positive probability.

Under what condition for parameters does the natural borrowing limit remain slack for a sufficiently long period? I argue below that the ratio ω_{min}/ω^e plays an important role. For brevity, let $a_0 = 0$. Then the right-hand side reduces to:

$$\frac{-a_{LB}/\omega^e}{1 - \omega_{min}/\omega^e} = \frac{\beta}{1-\beta}\frac{\omega_{min}/\omega^e}{1 - \omega_{min}/\omega^e}. \tag{18}$$

[7] In other words, $\omega_t = \epsilon\omega_{min} + (1-\epsilon)\mathbb{E}[\gamma_t | \gamma_t > \omega_{min}]$.

If we consider an annual model where the preference discount rate is 2%, $\beta/(1-\beta)$ is approximately 50. Then, for the natural borrowing limit not to bind for, say, 40 years, the ratio ω_{min}/ω^e needs to be as high as 0.44. If ω_{min}/ω^e is much lower and is, say, around 0.2, then it takes only about 13 periods for the natural borrowing limit to bind with strictly positive probability. Therefore, the ratio ω_{min}/ω^e needs to be sufficiently high for the linear solution not to violate the natural borrowing limit within a relatively short period.

Is 0.44 too high for a value of ω_{min}/ω^e? The answer, of course, depends on the distribution of γ_t. Here I examine two distributions for γ_t for which the ratio ω_{min}/ω^e can be analytically calculated. The first distribution is a Pareto distribution with parameter $\alpha > 1$. Then a closed-form expression for ω_{min}/ω^e can be obtained and is given by $\omega_{min}/\omega^e = (\alpha-1)/(\alpha-\epsilon)$.[8] If the Pareto parameter α equals 1.5, then $\omega_{min}/\omega^e \approx 0.44$ requires that $\epsilon \approx 0.364$. In other words, the income buffer protects the bottom 36% of the before-transfer income distribution. The second distribution to examine is a uniform distribution between 0 and 1. For $\omega_{min} \in [0,1]$, $\epsilon = \text{Prob}(\gamma_t \leq \omega_{min}) = \omega_{min}$ and the mean of ω_t is given by $w^e = (1+\omega_{min}^2)/2$. Then $\omega_{min}/\omega^e \approx 0.44$ requires that $\omega_{min} = \epsilon \approx 0.234$. Hence, the income buffer protects the bottom 23% of the before-transfer income distribution.

Note that these two distributions are chosen only to obtain some reference points. If the distribution of γ_t has an upper tail that diminishes more quickly than the Pareto and uniform distributions, the income buffer may need to protect a much smaller fraction of households at the bottom of the before-transfer income distribution. For such distributions, it might be possible to achieve simultaneously two goals: (1) the natural borrowing limit does not bind for a sufficiently long period with probability one, and (2) not too many households are protected by the income buffer.

5.2. *A model with large, persistent income loss*

Now I parameterize the flow income process to capture income risk associated with unemployment. The only source of uncertainty is idiosyncratic labor market risk. Each household receives the same labor income, w, if employed. Once separated, households start receiving the unemployment benefit, b. Each unemployed household may find a job with an exogenous job-finding rate, depending on the unemployment spell. The unemployment benefit duration is finite and, after a household remains unemployed for a given number of periods, the household receives no flow income during the rest of the unemployment spell.

Calibration. I pick values for some parameters from Krusell, Mukoyama, and Şahin (2010): one model period is set to six weeks, the preference discount factor β is set to 0.995, and the separation rate is set to 0.05.

There are some deviations from Krusell, Mukoyama, and Şahin (2010). Because Krusell, Mukoyama, and Şahin (2010) considered an incomplete-market general equilibrium model with labor market search and matching, both the real interest rate and the job-finding rate are endogenous in their model. Here, I focus only on the household's problem, and set the real interest rate and the job-finding rate exogenously. The gross real interest rate R is set to $1/\beta$. I defer the description of how the job-finding rate is calibrated.

Unlike Krusell, Mukoyama, and Şahin (2010), I assume that the duration of unemployment benefits is finite and that the job-finding rate varies with the unemployment spell. Because the unemployment benefits duration is up to 26 weeks for most states in the U.S., I assume that the unemployed households can collect the unemployment benefits for four consecutive periods (24 weeks) after losing a job cannot collect benefits from the fifth consecutive period of unemployment. The replacement rate is set to 0.4, that is, $b = 0.4w$. The wage w is assumed to be the same across workers and is, without loss of generality, normalized to one.[9]

To model a finite unemployment benefit duration, I assume that there are six possibilities for the individual household's labor market status: employed, unemployed for j periods where $j = 1, 2, \cdots 4$, and unemployed for more than or equal to five periods. To put it succinctly, the employment status s is drawn

[8] This is because $\mathbb{E}[\gamma_t | \gamma_t \geq \omega_{min}] = \alpha\omega_{min}/(\alpha-1)$.

[9] This is another point of departure from Krusell, Mukoyama, and Şahin (2010). In their general equilibrium model, households with different asset levels receive different wages because wages are determined via Nash bargaining and households' threat points depend on their asset levels.

from the set $S = \{e, u^1, u^2, \cdots, u^4, u^5\}$. Flow income ω is determined solely by s, and satisfies $\omega(e) = w$, $\omega(u^j) = b$ for $j = 1, 2, \cdots 4$, and $\omega(u^5) = 0$.

The duration-specific job-finding rates are computed using the estimated duration dependence in Hobijn and Sahin (2009). According to Hobijn and Sahin (2009), the estimated function for the U.S. is $f(m) = 0.755 \times \exp(-0.1158m)$, where m denotes the unemployment spell (in months) and $f(m)$ is the job-finding rate when the unemployment spell is m months. Because one model period is approximately 1.4 months, I compute the job-finding rates $f_j := f(1.4 \times j)$, where $j = 1, 2, \cdots, 4$ is the unemployment spell in model periods, and use these rates as transition probabilities to $s = e$ from $s = u^j$ for $j = 1, 2, \cdots, 4$. With the remaining probability $1 - f_j$, the state transits from u^j to u^{j+1} for $j = 1, 2, \cdots, 4$. For simplicity, the job-finding rate in state $s = u^5$ is also set to $f_5 := f(1.4 \times 5)$. Note, however, that in state u^5, unlike for $j \leq 4$, the state stays the same, with the remaining probability $1 - f_5$. In conjunction with the exogenous separation rate of 0.05, these numbers determine the whole transition matrix of the exogenous state.

Later I vary both the replacement rate and the benefit duration to examine how these parameters affect the probability of the borrowing constraints to bind.

Baseline results. The main exercise is to solve and simulate the model using the linear closed-form solution in Farmer (1990), and evaluate how likely the natural borrowing limit is to be violated. To this end, I simulate the model for $N = 50000$ households and 1000 periods. The initial distribution of the exogenous state s is set to its stationary distribution. The initial asset level a_0 is set to zero.

The natural borrowing limit depends on the current employment status but neither on time nor on the history of shocks. It satisfies

$$a^{LB}(s) = \max_{s' \in J(s)} \frac{a^{LB}(s') - \omega(s')}{R},$$

where $J(s)$ is a set of s' such that the transition probability from s to s' is strictly positive. The natural borrowing limit is zero for $s = u^4$ and $s = u^5$, because it is possible that the next period state is $s' = u^5$ for which the flow income is zero, and that the state stays there forever. In contrast, for other states, $s \in \{e, u^1, u^2, u^3\}$, it is negative because it is impossible to switch immediately to u^5, and the household in these states is guaranteed to receive some flow income with certainty in the future.

In the simulated data, a sizable fraction of households violated the natural borrowing limit. Figure 1 shows the time series of the fraction of households whose savings are below the natural borrowing limit. Initially, this fraction is close to zero but increases over time to around 25% within 10 years, 30% within 20 years, and 33% within 30 years.

To understand why this pattern emerges, it is important to note that when the underlying exogenous state s evolves as a first-order Markov process and when ω_t depends on the exogenous state only through the current s_t as $\omega_t = \omega(s_t)$, equation (10) implies that Δa_{t+1} is pinned down solely by s_t. Moreover, in the baseline specification, Δa is positive if and only if the current state is $s = e$. In all other states of unemployment, savings decrease.

Figure 2 depicts the time series of the fraction of households whose savings are below the natural borrowing limit, conditional on the current state. Because some households transit to the employment state from unemployment states in which savings decrease, some have already violated the natural borrowing limit. Even though they increase savings in state $s = e$, the increase may not be enough to lift the savings above the natural borrowing limit. This is the reason why the share keeps increasing in state $s = e$. The same pattern is also observed in state $s = u^1$, simply because some households transit directly from $s = e$ to $s = u^1$ when separated. For states $s \in \{u^3, u^4, u^5\}$, the opposite pattern is observed. Initially most of the households in these states violate the natural borrowing limit because they start with zero savings and their savings decrease in these states. But, over time, some households who have accumulated positive savings in state $s = e$ become unemployed and transit to these states, thereby lowering the share of households who violate the natural borrowing limit.

Figure 3 shows, for each state, the histogram of savings in selected time periods, together with the natural borrowing limit (a red dotted line). It is clear that, in all states, a sizable fraction of households violates the natural borrowing limit.

Results with a different replacement rate. If the replacement rate is raised, the natural borrowing limit is relaxed and, thus, is less likely to bind. However, I still find that a sizable fraction of households violates the natural borrowing limit even when the replacement rate is set to a high value. For example, if I double the replacement rate to 0.8, the fraction of households that violate the natural borrowing limit is around 8% after 10 years, 13% after 20 years, and 16% after 30 years (Figure 4a). These numbers are about a half of those found under the replacement rate of 0.4, but definitely not negligible.

Results with a different benefit duration. Another parameter that is likely to be important is the unemployment benefits duration because the duration, together with the replacement rate, determines the natural borrowing limit. For example, the natural borrowing limit for the employed in the baseline model equals the present value of the unemployment benefits for the following four periods (= maximal duration of benefits). Therefore, if the duration is extended from four periods, the natural borrowing limit is relaxed and is less likely to bind.

To examine how this parameter affects the probability that the natural borrowing limit is violated, I also simulated the model using a longer unemployment benefit duration of nine periods (=54 weeks), roughly twice as long as the actual maximal duration of 26 weeks in the U.S.

The result is that doubling the unemployment benefit duration has a smaller effect than doubling the replacement rate. Figure 4b plots the share of households that violate the natural borrowing limit for each period t. The share is higher than the share in Figure 4a. Again, these numbers are smaller than those under the baseline calibration but, nevertheless, not negligible.

6. Conclusion

This paper has reexamined whether the consumption-saving problem under the RINCE preferences implies linear decision rules for consumption and savings. Whether the linearity holds or not is important because it has been exploited to facilitate aggregation in macroeconomic models. I have pointed out that the proof of linearity in Farmer (1990) implicitly assumes that the natural borrowing constraint never binds, and I have provided a counterexample in which the linear decision rule in Farmer (1990) indeed violates the natural borrowing limit and therefore is not a correct solution to the problem. I have used two models that feature some realistic characteristics of income risk to demonstrate that the linear decision rule in Farmer (1990) violates the natural borrowing limit with strictly positive probability, and that the probability can indeed be sizable.

Given these findings, one needs to be cautious about using the RINCE preferences to exploit the linearity of a solution to facilitate aggregation in incomplete-market macroeconomic models. Unless one sufficiently restricts the household's problem structure so that the natural borrowing limit never binds, one may likely use an incorrect solution to obtain the wrong aggregation results.

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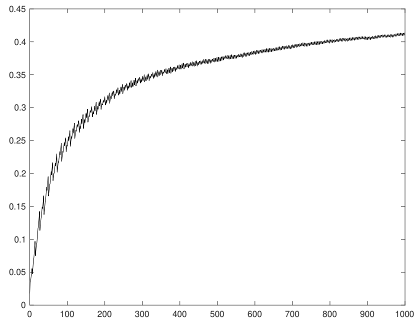
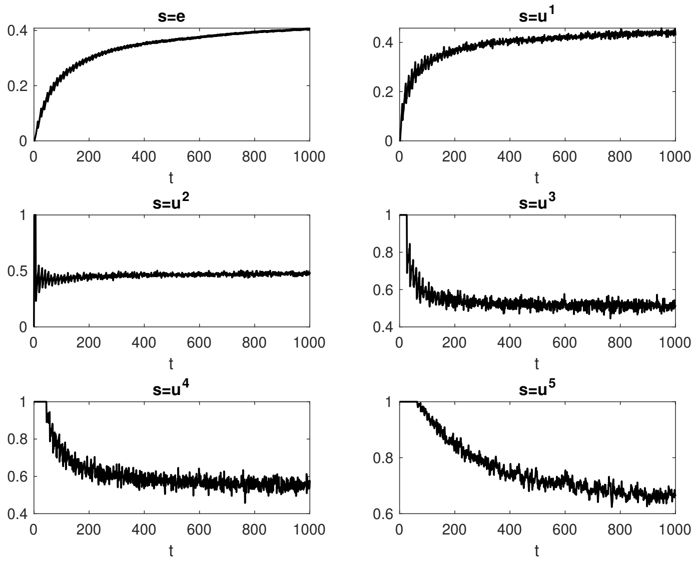


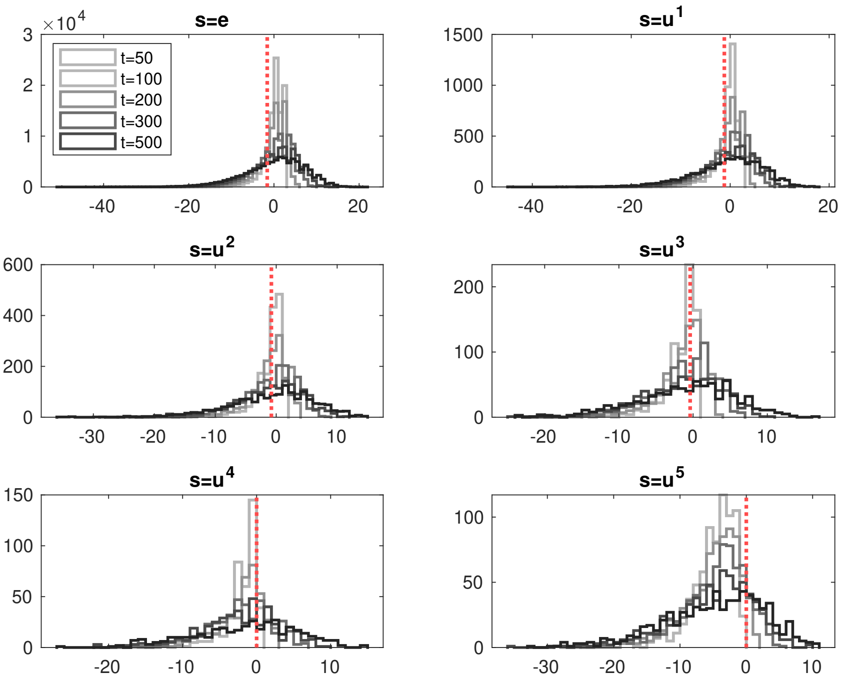
Dynamics of the share of households that violate the natural borrowing limit at each point in time are depicted. Time t is on the horizontal axis and the share of households that violate the natural borrowing limit at each point in time is on the vertical axis.

Figure 1: Share of observations that violate the natural borrowing limit



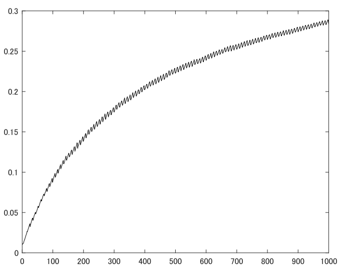
Dynamics of the share of households that violate the natural borrowing limit at each point in time for each state are depicted. In each panel, time t is on the horizontal axis and the share is on the vertical axis.

Figure 2: Share of observations that violate the natural borrowing limit, conditional on the current state

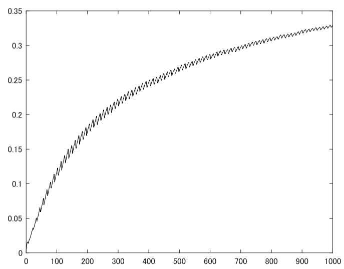


In each panel, the savings histograms for the corresponding state are shown for selected time periods. The natural borrowing limit is indicated by a vertical, dotted line.

Figure 3: Savings histograms conditional on the current state



(a) Replacement rate = 0.8



(b) Duration = nine periods

Panel (a) shows the dynamics for the replacement rate of 0.8. Panel (b) shows the dynamics for the unemployment benefit duration of nine periods. In each panel, time *t* is on the horizontal axis and the share of households that violate the natural borrowing limit at each point in time is on the vertical axis.

Figure 4: Share of observations that violate the natural borrowing limit